

Large Quantum Gravity Effects: Cylindrical Waves in Four Dimensions

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February 7, 2008

Abstract

Linearly polarized cylindrical waves in four-dimensional vacuum gravity are mathematically equivalent to rotationally symmetric gravity coupled to a Maxwell (or Klein-Gordon) field in three dimensions. The quantization of this latter system was performed by Ashtekar and Pierri in a recent work. Employing that quantization, we obtain here a complete quantum theory which describes the four-dimensional geometry of the Einstein-Rosen waves. In particular, we construct regularized operators to represent the metric. It is shown that the results achieved by Ashtekar about the existence of important quantum gravity effects in the Einstein-Maxwell system at large distances from the symmetry axis continue to be valid from a four-dimensional point of view. The only significant difference is that, in order to admit an approximate classical description in the asymptotic region, states that are coherent in the Maxwell field need not contain a large number of photons anymore. We also analyze the metric fluctuations on the symmetry axis and argue that they are generally relevant for all of the coherent states.

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1 Introduction

The quantization of spacetimes with two commuting spacelike Killing fields has deserved intensive study during recent years [1-11]. The main reason for this interest is the ability of this type of spacetimes to provide a suitable framework in which one can discuss conceptual problems and develop mathematical methods for the quantization of general relativity. The families of solutions with two Killing fields that have been quantized in the literature, though simple enough as to be tractable, still possess an infinite number of degrees of freedom, so that they are expected to retain the field complexity that should be present in the elusive theory of quantum gravity.

Additional motivation for the quantum analysis of spacetimes with two commuting Killing fields comes from their possible application to cosmology and astrophysics. Most of the spacetimes of this kind that have been subject to quantization can in fact be interpreted as gravitational waves propagating in a source-free background. Particular examples are the Gowdy cosmologies with the spatial topology of the three-torus [8, 12], the family of purely gravitational plane waves [9, 10, 13], and the set of cylindrical waves in vacuum gravity [1, 2]. Among all of them, it is the family of cylindrical waves whose quantization has received more attention and is probably best understood [1-7,14,15].

The pioneer of this midisuperspace approach to quantum gravity was Kuchař [14]. He made a preliminary discussion of the quantum mechanics for Einstein-Rosen waves [16], i.e., cylindrical gravitational waves in four dimensions with linear polarization. The classical description of these waves is equivalent to that corresponding to a rotationally symmetric massless scalar field coupled to three-dimensional gravity [14, 17] and, therefore, to the description of a rotationally symmetric Einstein-Maxwell model in three dimensions [4]. Recently, a rigorous quantization of this three-dimensional counterpart of the Einstein-Rosen model has been carried out by Ashtekar and Pierri [1], superseding previous work on the subject by Allen [15]. Some technical details concerning the self-adjointness of the metric operators in this three-dimensional system have been revised by Varadarajan [3]. On the other hand, a quantum theory for the most general family of cylindrical waves in four-dimensional gravity, which exploits the group-theoretical properties of the system, has been presented by Korotkin and Samtleben, although no explicit construction has been provided for the metric operators [2].

An issue that has been investigated with special interest in this context

is the existence of quantum gravitational states that can be approximated by a classical solution, or semiclassical one if quantum matter fields are included. By analyzing the three-dimensional theory that is obtained from the Einstein-Rosen waves via dimensional reduction, Ashtekar has proved that, at least in a certain sector of quantum gravity, the semiclassical approximation may become meaningless owing to the appearance of huge quantum gravity effects [4]. Namely, in the rotationally symmetric Einstein-Maxwell model in three dimensions, Ashtekar has considered all coherent states of the Maxwell field and computed their expectation values and quantum fluctuations in the three-metric at large distances from the center (i.e., the axis of symmetry). For coherent states that are sharply peaked around a characteristic wave number k_0 , the asymptotic expectation value of the three-metric is peaked around a classical solution if and only if $N(e^{k_0} - 1)^2 \ll 1$, where N is the number of photons contained in the state. In addition, if one requires the quantum uncertainties in the Maxwell field to be relatively small, the semiclassical description is accurate only when $N \gg 1$ [4]. Here, and in the rest of the paper, we have used units in which $c = \hbar = 8G_3 = 1$, G_3 being the gravitational constant in three dimensions or, equivalently, the effective Newton constant per unit length in the direction of the symmetry axis.

The possibility of finding states in the Einstein-Maxwell system with improved coherence in the three-metric at the expense of increasing the dispersion in the Maxwell field was proved by Gambini and Pullin [5]. Large quantum gravity effects similar to those detected by Ashtekar were also found in the rotationally symmetric Einstein-Maxwell model by employing non-local variables [6], and in a three-dimensional model with toroidal symmetry [11]. Only one purely four-dimensional gravitational system has been discussed in which the quantum fluctuations invalidate the classical description of the geometry: a midisuperspace model for linearly polarized plane waves in vacuum gravity [9]. In this case, the huge fluctuations appear in a region where null geodesics are focused, and not in the asymptotic region.

The aim of the present work is to revisit Ashtekar's results about the existence of large quantum effects in cylindrical gravity from a strictly four-dimensional point of view. The classical equivalence of the Einstein-Rosen and the three-dimensional Einstein-Maxwell systems does not necessarily imply their quantum equivalence. On the other hand, since the Einstein-Rosen model and its three-dimensional counterpart have different metrics, all questions about the existence of quantum states peaked around classical geometries in general relativity should be addressed from a four-dimensional per-

spective. In fact, as we will see, the Einstein-Rosen metric can be expressed as a function of the Maxwell field and the metric in three dimensions which is highly non-linear in the matter field. As a consequence, coherence in the four-metric does not generally follow from coherence in the three-metric and the field.

The plan of the paper is the following. In Sec. 2, we construct a midisuperspace model for cylindrical waves starting from the Hamiltonian formulation of general relativity for spacetimes that possess two commuting spacelike Killing fields. We adopt a gauge-fixing procedure that is similar to that introduced by Ashtekar and Pierri in three dimensions [1], and calculate the reduced Hamiltonian of the model by a careful analysis of surface terms in the gravitational action. This framework is then particularized to the case of linearly polarized waves via symmetry reduction. Employing the quantum theory put forward in Ref. [1] for the rotationally symmetric three-dimensional system, we present a complete quantum theory for the Einstein-Rosen waves in Sec. 3. In particular, we obtain regularized, positive operators to describe all components of the four-metric. The behavior of these operators on the quantum states that are coherent in the Maxwell field is discussed in Sec. 4. We first analyze the quantum gravitational effects at large distances of the symmetry axis, showing that the conclusions reached by Ashtekar for the three-dimensional metric are valid as well for the metric of the Einstein-Rosen waves, not only qualitatively, but also quantitatively. From the point of view of the four-metric, however, the requirement of a classical behavior for the Maxwell field is now spurious, so that the condition $N \gg 1$ is no longer necessary to reach an acceptable classical approximation in the asymptotic region. Using our four-dimensional formalism, we are also able to study the quantum fluctuations in the metric on the symmetry axis. We argue that these fluctuations cannot be neglected for any of the coherent states. We summarize our results and conclude in Sec. 5. Finally, two appendices are added. In Appendix A we prove some useful operator identities, while Appendix B contains some calculations employed in the discussion of the metric fluctuations.

2 The Midisuperspace Model

Let us first construct a gauge-fixed midisuperspace model to describe cylindrical waves in vacuum gravity. Since this family of waves can be regarded as

a particular class of spacetimes that possess two commuting Killing vectors, we can start our analysis with the Hamiltonian formulation for spacetimes of this kind, which is discussed in Sec. 3 of Ref. [10]. For convenience, we adopt the notation $\{x^i\} \equiv \{Z, \theta, R\}$ ($i = 1, 2, 3$) for the spatial coordinates and assume that the two commuting Killing vector fields are ∂_{x^a} ($a = 1, 2$), so that the metric is independent of θ and Z . In addition, we impose that $R \geq 0$ and $\theta \in S^1$ (with S^1 being the unit circle). With this terminology, Z denotes the coordinate of the symmetry axis, whereas R and θ are the radial and angular coordinates on each surface of constant Z and time t .

The momentum constraints corresponding to the coordinates x^a can be eliminated by requiring that the induced three-metric h_{ij} is block-diagonal, namely, $h_{aR} = 0$ (where we have adopted the alternative notation h_{aR} instead of h_{a3}). The gauge fixing is almost identical to that explained in Ref. [10] for the case of plane waves, and we will not repeat details here. Apart from the different domains of definition for the spatial coordinates, the only modification that must be introduced concerns the system of units. In the cited paper, the authors set $c = \hbar = 4G_3 = 1$, where $G_3 = G/(\int dZ)$ is the effective Newton constant per unit length. In the present work, however, we have fixed $8G_3 = 1$ (to facilitate comparison of our results with those of Ashtekar and Pierri). We can nevertheless take account of this discrepancy by simply multiplying all gravitational constraints in Ref. [10] by a factor of two and dividing the canonical momenta of the metric functions by the same factor.

As shown in Ref. [10], the dynamical stability of the gauge-fixing conditions $h_{aR} = 0$ requires that $4N\sqrt{h_{RR}}f_a = -\sqrt{\det h_{cd}}h_{ab}(N^b)'$, where f_a are two constants (independent of R) that determine the momenta of h_{aR} . Here, N is the lapse function, N^i is the shift vector, and the prime stands for the derivative with respect to R . Since the two-metric h_{ab} becomes degenerate on the symmetry axis (that we suppose located at $R = 0$), the regularity of the four-metric on this axis implies that the constants f_a must vanish. As a consequence, we conclude that the components N^a of the shift vector are independent of the spatial coordinates, and can be absorbed by a redefinition of x^a . It hence turns out that the condition of regularity on the axis suffices to ensure that the orbits spanned by the two Killing vectors admit orthogonal surfaces.

The remaining momentum constraint can be eliminated in a very similar way to that discussed at the end of Sec. 3 and the beginning of Sec. 4 in Ref. [10]. One only needs to change the choice of the strictly increasing

function z_0 that determines the coordinate R . We now select $z_0 = \ln R$. In this way, the radial coordinate R is set to coincide with the square root of the determinant of the metric on Killing orbits. We notice that our gauge fixing for the momentum constraint associated with R is analogous to that performed by Ashtekar and Pierri in three dimensions [1].

The resulting reduced system has a configuration space with three degrees of freedom which, with the conventions of Ref. [10], can be chosen as the three metric functions v , y , and w . In order to adopt a notation similar to that employed in the three-dimensional Einstein-Maxwell model [1, 4], it is convenient to introduce the definitions $\psi = \ln R - y/2$ and $\gamma = 2w$. The system has still one constraint, namely, the Hamiltonian constraint, which can now be written [10]

$$\begin{aligned} \mathcal{H} = & \frac{e^{(\psi-\gamma)/2}}{2R} \left[R^2(\psi')^2 - 2R\gamma' + e^{2\psi}(v')^2 + p_\psi^2 + R^2 e^{-2\psi} p_v^2 \right] \\ & + e^{(\psi-\gamma)/2} p_\gamma (p_v v' + p_\psi \psi' + p_\gamma \gamma' - 2p'_\gamma). \end{aligned} \quad (2.1)$$

The corresponding gauge freedom can be eliminated by imposing the vanishing of the momentum canonically conjugate to γ : $p_\gamma = 0$. This condition is inspired by the gauge fixing carried out in the three-dimensional counterpart of our model [1]. It is straightforward to check that the gauge fixing is well posed. In addition, the gauge condition is preserved by the dynamical evolution provided that $\{p_\gamma, \int dR N \mathcal{H}\} \doteq -(e^{(\psi-\gamma)/2} N)' = 0$, where the symbols $\{, \}$ and \doteq denote Poisson brackets and weak identity, respectively. Hence, the lapse function must be of the form $N = f(t)e^{(\gamma-\psi)/2}$, with $f(t)$ being a function of time (that can generally be absorbed by a redefinition of t). We will choose this function equal to $e^{-\gamma_\infty/2}$, where γ_∞ is the value of the metric function γ when $R \rightarrow \infty$. As we will see below, this choice guarantees that ∂_t is a unit asymptotic time translation.

On the other hand, the solution to the Hamiltonian constraint with $p_\gamma = 0$ is

$$\gamma = \frac{1}{2} \int_0^R d\bar{R} \bar{R} \left[(\psi')^2 + \frac{p_\psi^2}{\bar{R}^2} + e^{2\psi} \frac{(v')^2}{\bar{R}^2} + e^{-2\psi} p_v^2 \right], \quad (2.2)$$

where we have imposed that γ vanish at $R = 0$ in order to obtain (with suitable boundary conditions on ψ and v) a regular metric on the axis of symmetry. After our gauge fixing, the line element has the expression

$$ds^2 = e^{-\psi} \left[e^\gamma (-e^{-\gamma_\infty} dt^2 + dR^2) + R^2 d\theta^2 \right] + e^\psi (dZ - v d\theta)^2. \quad (2.3)$$

Assuming as a boundary condition (see Ref. [1] for a detailed discussion in the linearly polarized case) that the metric functions ψ and v fall off sufficiently fast as $R \rightarrow \infty$ (so that, in particular, γ_∞ is finite), we get that the above metric describes an asymptotically flat spacetime with a generally non-zero deficit angle. In this asymptotic region, as we anticipated, ∂_t is a unit timelike vector.

The reduced model obtained in this way is free of constraints and has only two metric degrees of freedom, described by the variables ψ and v . Its reduced symplectic structure is $\Omega = \int dR(\mathbf{d}p_\psi \wedge \mathbf{d}\psi + \mathbf{d}p_v \wedge \mathbf{d}v)$, where \mathbf{d} and \wedge denote, respectively, the exterior derivative and product. The Hamiltonian that generates the dynamics of the model, on the other hand, can be obtained by reducing the gravitational Hilbert-Einstein action supplemented with appropriate boundary terms [18]. Let us explain this point in more detail. In our gauge-fixing procedure, we have removed some of the original degrees of freedom by expressing them in terms of the remaining canonical variables and, possibly, of the coordinates. All the expressions employed are in fact local, except in the very last step of the procedure, where relation (2.2) has been introduced. It is not difficult to realize then that, in our discussion of the gauge fixing, the dynamical equations that we have computed via Poisson brackets are actually valid in the interior of our manifold, even though we have not explicitly included surface terms in the Hamiltonian. This fact ensures that our gauge fixing has been carried out consistently. Furthermore, it then follows that the Hamiltonian of the reduced model is actually given by the reduction of the total Hamiltonian (including surface terms) of our original system. Since, as we have pointed out, relation (2.2) is not local, this reduced Hamiltonian may be non-trivial.

The boundary terms for the gravitational Hamiltonian have been recently analyzed by Hawking and Hunter [18]. To apply their results to our reduced model, let us first consider a manifold that, on each section Σ_t of constant time, has a two-dimensional boundary B_t which is a cylinder of radius R_f [19]. In addition, we assume that the spacetime metric has the form (2.3). Then, in the limit $R_f \rightarrow \infty$ we clearly reach the family of cylindrical waves that we want to study. Since all the constraints have been eliminated in the process of gauge fixing and the shift vector vanishes in Eq. (2.3), it is not difficult to conclude from the discussion in Ref. [18] that the Hamiltonian of our reduced model comes exclusively from boundary terms on B_t , and is

given by

$$H = - \lim_{R_f \rightarrow \infty} 2N\sqrt{\sigma}(\kappa - \kappa_0). \quad (2.4)$$

Here, we have made $8G_3 = 1$, σ is the determinant of the two-metric induced on B_t , and κ and κ_0 are the trace of the extrinsic curvature of this metric embedded, respectively, in Σ_t and in a three-dimensional Minkowski background. It is straightforward to check that $\kappa = e^{-\gamma_\infty/2}/(NR_f)$ and $\kappa_0 = 1/R_f$, while $\sigma = R_f^2$. Therefore, we obtain that the reduced Hamiltonian that generates time evolution in the coordinate t is $H = 2(1 - e^{-\gamma_\infty/2})$. In particular, this implies that γ_∞ is a constant of motion, because it commutes with the Hamiltonian. So, given any classical solution, it is possible to absorb the factor $e^{-\gamma_\infty}$ in the line element by a mere rescaling of the time coordinate: $T = e^{-\gamma_\infty/2}t$ (off-shell, one would have $T = \int_0^t d\bar{t} e^{-\gamma_\infty/2}$).

Let us now particularize our considerations to the simpler case of linearly polarized cylindrical waves. For the Einstein-Rosen waves, we have $v = 0$. We can impose this restriction as a symmetry condition in our model. Its compatibility with the Hamiltonian evolution leads to the secondary constraint $p_v = 0$. One can also check that there are not tertiary constraints. The symmetry conditions $v = p_v = 0$ form a pair of second-class constraints that allow the reduction of the model by removing two canonical degrees of freedom. The resulting system has the following metric and symplectic structure:

$$ds^2 = e^{-\psi}[e^\gamma(-dT^2 + dR^2) + R^2 d\theta^2] + e^\psi dZ^2, \quad (2.5)$$

$$\gamma = \frac{1}{2} \int_0^R d\bar{R} \bar{R} \left[(\psi')^2 + \frac{p_\psi^2}{\bar{R}^2} \right], \quad (2.6)$$

$$\Omega = \int_0^\infty dR \, \mathbf{d}p_\psi \wedge \mathbf{d}\psi. \quad (2.7)$$

Note that the term between square brackets in Eq.(2.5) is precisely the gauge-fixed metric of the dimensionally reduced Einstein-Maxwell model discussed in Ref. [1]. In addition, the reduced Hamiltonian coincides also with that found by Ashtekar and Pierri in three-dimensions. Finally, it is straightforward to see that, in terms of the time coordinate T , the dynamical equations for the field ψ are exactly those satisfied by a rotationally symmetric massless scalar field in three-dimensional Minkowski spacetime [1]. This scalar field can be interpreted as the dual of a Maxwell field [4]. In this way, one recovers the Einstein-Maxwell analog in three dimensions of the Einstein-Rosen waves.

3 Quantum Theory

Since the field ψ is a rotationally symmetric solution to the massless Klein-Gordon equation in three dimensions that is regular at the origin $R = 0$ [1], all classical solutions admit the mode expansion

$$\psi(R, T) = \frac{1}{\sqrt{2}} \int_0^\infty dk J_0(kR) \left[A(k) e^{-ikT} + A^\dagger(k) e^{ikT} \right]. \quad (3.1)$$

The constants of motion $A(k)$ and $A^\dagger(k)$ are complex conjugate to each other, because ψ and J_0 (i.e., the zeroth-order Bessel function of the first kind) are real. Employing the identity $2\pi J_0(kR) = \oint d\theta e^{ikR \cos \theta}$, we can write the above expression in the alternative form

$$\psi(R, T) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{d^2 k}{\sqrt{2} |\vec{k}|} \left[A(|\vec{k}|) e^{i(\vec{k} \cdot \vec{x} - |\vec{k}|T)} + A^\dagger(|\vec{k}|) e^{-i(\vec{k} \cdot \vec{x} - |\vec{k}|T)} \right], \quad (3.2)$$

with $R = |\vec{x}|$. Taking then into account that, from the Hamiltonian equations of motion, $p_\psi = R\dot{\psi}$, where the overdot stands for the derivative with respect to T , substitution of Eq. (3.2) in the symplectic form leads to $\Omega = i \int_0^\infty \mathbf{d}A^\dagger(k) \wedge \mathbf{d}A(k)$. Therefore, $A(k)$ and $A^\dagger(k)$ can be understood as annihilation and creation like variables. In addition, a trivial calculation using Eqs. (2.6) and (3.2) shows that γ_∞ equals the Hamiltonian of the massless scalar field [1]: $\gamma_\infty = \int_0^\infty dk k A^\dagger(k) A(k)$.

Essentially, the quantization of our basic field ψ can then be carried out by introducing a Fock space in which $\psi(R, T)$ goes over to an operator-valued distribution $\hat{\psi}(R, T)$, obtained by representing $A(k)$ and $A^\dagger(k)$ as standard annihilation and creation operators [1, 3, 15]. The Fock space in this representation is that over the Hilbert space of square integrable functions on the positive real axis, $L^2(\mathbb{R}^+, dk)$. Using such a representation, a complete quantization of the Einstein-Maxwell counterpart of our system has been recently proposed [1, 3]. Our aim in this section is to show how the quantization put forward by Ashtekar and Pierri in three dimensions can be employed to construct a consistent quantum theory which fully describes the four-dimensional metric of the Einstein-Rosen model.

As a first step towards the introduction of meaningful metric operators, let us regularize the basic field $\hat{\psi}(R, T)$, which is defined only as an operator-valued distribution [the reason being that $J_0(kR)$ does not belong to $L^2(\mathbb{R}^+, dk)$ for any $R \geq 0$]. Given that J_0 is bounded in \mathbb{R}^+ , the regularization can be achieved by simply multiplying the factor $J_0(kR)$ in the quantum

version of Eq. (3.1) by a square integrable real function, $g \in L^2(\mathbb{R}^+, dk)$,

$$\hat{\psi}(R, T|g) = \frac{1}{\sqrt{2}} \int_0^\infty dk J_0(kR) g(k) [\hat{A}(k) e^{-ikT} + \hat{A}^\dagger(k) e^{ikT}]. \quad (3.3)$$

This regularization can be justified from a physical point of view, e.g., by admitting the existence of a cut-off k_c in momentum space [15]. The corresponding function $g(k)$ equals then the unity on the compact interval $[0, k_c]$ and vanishes outside. In this sense, it is worth pointing out that the model itself provides an energy scale, namely, c^4/G_3 (adopting a general system of units). Thus, a natural candidate for k_c could be $c^3/(\hbar G_3)$, which has dimensions of an inverse length.

As an alternative motivation for the regularization, one can just smear the operator-valued distribution $\hat{\psi}(\vec{x}, T) \equiv \hat{\psi}(R = |\vec{x}|, T)$, defined via the quantum analog of Eq. (3.2), with a test function in two dimensions $f(\vec{x})$ that is also rotationally symmetric. We assume that $f(\vec{x})$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of smooth functions on the plane with rapid decrease at infinity. In order to interpret the smearing as an average, we further accept that $f(\vec{x})$ is real and has a unit integral over \mathbb{R}^2 . Then, a simple calculation proves that the smeared operator $\int d^2x_0 f(\vec{x}_0) \hat{\psi}(\vec{x} - \vec{x}_0)$ is rotationally symmetric and can be expressed in the form (3.3), with $g(k)$ given by

$$2\pi \tilde{f}(k) = \int_{\mathbb{R}^2} d^2x f(\vec{x}) e^{i\vec{k} \cdot \vec{x}} = 2\pi \int_0^\infty dR R J_0(kR) f(R). \quad (3.4)$$

Here, $\tilde{f}(\vec{k})$ denotes the Fourier transform of $f(\vec{x})$ in two dimensions, and the notation $\tilde{f}(k)$ and $f(R)$ indicates that these functions depend only on $k = |\vec{k}|$ and $R = |\vec{x}|$, respectively. The last identity in the above equation shows that $\tilde{f}(k)$ is real; hence, so is $g(k)$. The first identity, together with the properties of the Fourier transform [20] and the fact that $f(\vec{x})$ belongs to the Schwartz space, implies that $\tilde{f}(\vec{k}) \in \mathcal{S}(\mathbb{R}^2)$. Then, we have that $g(k)$ belongs to the Hilbert space $L^2(\mathbb{R}^+, dk)$. In addition, since $f(\vec{x})$ has unit integral, it follows that $g(0) = 2\pi \tilde{f}(0) = 1$.

For any choice of the real function $g \in L^2(\mathbb{R}^+, dk)$, the operator (3.3), with domain given by the dense subspace of the Fock space consisting of all finite particle vectors [20], is symmetric and admits a self-adjoint extension [20], which we will denote again with the symbol $\hat{\psi}(R, T|g)$. The standard spectral theorems ensure then that the exponential operators $e^{\pm \hat{\psi}(R, T|g)}$ are well-defined and positive. Besides, recalling the definition of normal ordering

and the Campbell-Baker-Hausdorff (CBH) formula $e^{\hat{a}}e^{\hat{b}} = e^{[\hat{a}, \hat{b}]/2}e^{\hat{a}+\hat{b}}$, which is valid for operators whose commutator is a c -number [21], we conclude

$$: e^{\pm \hat{\psi}(R, T|g)} := e^{-||g_R||^2/2} e^{\pm \hat{\psi}(R, T|g)}, \quad (3.5)$$

where $|| \cdot ||$ denotes the norm in $L^2(\mathbb{R}^+, dk)$ and

$$g_R(k) = \frac{1}{\sqrt{2}} J_0(kR) g(k). \quad (3.6)$$

The diagonal θ and Z components of the four-metric can then be represented by the positive operators

$$\hat{h}_{\theta\theta}(R, T|g) = R^2 : e^{-\hat{\psi}(R, T|g)} : , \quad \hat{h}_{ZZ}(R, T|g) = : e^{\hat{\psi}(R, T|g)} : \quad (3.7)$$

Note that the normal ordering in these definitions guarantees that the vacuum expectation values reproduce the classical values of $h_{\theta\theta}$ and h_{ZZ} in Minkowski spacetime.

On the other hand, the representation of the metric function (2.6) by a regularized operator $: \hat{\gamma}(f_R, T) :$ was discussed in Refs. [1, 3]. The symbol f_R denotes a smearing function employed in the regularization, namely, $f_R(r)$ is a function on the positive real axis that equals the unity for all $r \leq R$, decreases smoothly in $[R, R+\epsilon]$ and vanishes for $r \geq R+\epsilon$, with $\epsilon > 0$ being a certain parameter with dimensions of length [22]. It has been recently shown [3] that this regularized operator has a well-defined action on a dense subspace of the Fock space which is contained in the set of finite particle vectors. In that domain of definition, the operator $: \hat{\gamma}(f_R, T) :$ is symmetric [3]. As a straightforward consequence, so is $: \hat{\gamma}(f_R, T) : - \hat{\psi}(R, T|g)$ provided that the real function g belongs to $L^2(\mathbb{R}^+, dk)$. In addition, Varadarajan has argued that $: \hat{\gamma}(f_R, T) :$ admits a self-adjoint extension, because it is (formally) possible to find a conjugation [20] that leaves invariant the domain of definition of this operator and commutes with it [3]. In fact, the same argument supports the existence of a self-adjoint extension of $: \hat{\gamma}(f_R, T) : - \hat{\psi}(R, T|g)$, because it is easy to check that the considered conjugation commutes as well with $\hat{\psi}(R, T|g)$ when g is real. Using the spectral theorem, we would then conclude that the exponential of this self-adjoint extension,

$$\hat{\Gamma}(R, T|g, f_R) \equiv e^{:\hat{\gamma}(f_R, T) : - \hat{\psi}(R, T|g)}, \quad (3.8)$$

is a well-defined, positive operator.

We can then represent the remaining non-trivial components of the four-metric (i.e., the diagonal R and T components) by the operator

$$\hat{h}_{RR}(R, T|g, f_R) = e^{-\|\bar{g}_R\|^2} \hat{\Gamma}(R, T|g_1, f_R), \quad (3.9)$$

where we have adopted the notation

$$\bar{g}_R(k) = \frac{\sqrt{e^k - 1 - k}}{e^k - 1} g_R(k), \quad g_1(k) = \frac{k}{e^k - 1} g(k), \quad (3.10)$$

and used definition (3.6). It is readily seen that the functions \bar{g}_R and g_1 belong to $L^2(\mathbb{R}^+, dk)$ if so does the function g . According to our discussion above, the introduced operator should then be positive if the function g is real and square integrable on the positive axis. In our definition (3.9), the numerical factor $e^{-\|\bar{g}_R\|^2}$, as well as the replacement of g with g_1 as the regularization function used in $\hat{\Gamma}$, can be understood as a convenient choice of factor ordering. Indeed, after restoring the dimensional constants c , \hbar , and G_3 in our calculations, it is possible to check that, when $\hbar \rightarrow 0$, the factor $e^{-\|\bar{g}_R\|^2}$ tends to the unity, whereas $g_1 \rightarrow g$. The selected factor ordering is motivated by the following considerations.

In the limit $R \rightarrow \infty$, the smearing function f_R tends to the unit function, and the operator $:\hat{\gamma}(f_R, T):$ becomes

$$:\hat{\gamma}_\infty := \int_0^\infty dk \, k \hat{A}^\dagger(k) \hat{A}(k), \quad (3.11)$$

which is the normal ordered Hamiltonian of a rotationally symmetric, massless scalar field in three dimensions [1, 4]. We then obtain that, in the asymptotic region $R \rightarrow \infty$, the purely radial component of the quantum metric is given by $\lim_{\bar{R} \rightarrow \infty} \hat{h}_{RR}(\bar{R}, T|g, 1)$. On the other hand, it is shown in Appendix A that

$$\hat{h}_{RR}(\bar{R}, T|g, 1) = e^{-\int_0^\infty dk g_{\bar{R}}(k) e^{ikT} \hat{A}^\dagger(k)} e^{:\hat{\gamma}_\infty:} e^{-\int_0^\infty dk g_{\bar{R}}(k) e^{-ikT} \hat{A}(k)}. \quad (3.12)$$

Therefore, our factor ordering ensures that, at least in the asymptotic region, the vacuum expectation value of the metric operator (3.9) coincides with the classical value of h_{RR} in Minkowski spacetime, a value which is equal to the unity. In addition, the factor ordering adopted is also very convenient from a practical point of view, because, for polynomials of the operator (3.12), all matrix elements between coherent states of the basic field ψ are explicitly

computable. For such coherent states, one can then complete the calculation of the asymptotic fluctuations in \hat{h}_{RR} . Moreover, the operator $e^{i\hat{\gamma}_\infty}$ that appears in Eq. (3.12) is precisely the operator employed by Ashtekar and Pierri to represent the purely radial component of the metric in the three-dimensional counterpart of our system [1]. As we will see in the next section, this fact leads to a simple relation between the coherent expectation values and fluctuations obtained for the radial component of the asymptotic metric in the four and three-dimensional models.

4 Metric fluctuations

We are now in an adequate position to study the quantum geometry of the model and discuss whether the conclusions obtained by Ashtekar in three dimensions about the existence of large quantum gravity effects in the asymptotic region generalize to the four-dimensional model describing Einstein-Rosen waves. Like in the analysis of Ref. [4], we will only consider quantum states that are coherent in the basic field ψ . These states show the most classical behavior that is allowed for the fundamental field of the theory [21]. As such, they are natural candidates in the search for states that admit an approximate classical description of the geometry.

Given any complex function $C \in L^2(\mathbb{R}^+, dk)$, there exists an associated coherent state $|C\rangle$ of unit norm, which has the form

$$|C\rangle = e^{-\|C\|^2/2} e^{\int_0^\infty dk C(k) \hat{A}^\dagger(k)} |0\rangle. \quad (4.1)$$

Here, $|0\rangle$ is the unique vacuum of the Fock space. For any coherent state, the expectation value of the (regularized) field $\hat{\psi}(R, T|g)$ coincides, at all values of R and T , with the classical field solution obtained by replacing the annihilation and creation operators with the functions $C(k)$ and its complex conjugate:

$$\langle \hat{\psi}(R, T|g) \rangle_C = 2 \int_0^\infty dk g_R(k) \text{Re}[C(k) e^{-ikT}], \quad (4.2)$$

with $\text{Re}[\cdot]$ denoting the real part. In addition, using definitions (3.7) and the CBH formula, one can check that the coherent expectation values of the diagonal θ and Z components of the metric are also equal to the corresponding classical expressions:

$$\langle \hat{h}_{\theta\theta}(R, T|g) \rangle_C = R^2 \left(\langle \hat{h}_{ZZ}(R, T|g) \rangle_C \right)^{-1} = R^2 e^{-\langle \hat{\psi}(R, T|g) \rangle_C}. \quad (4.3)$$

The calculation of the expectation value of the purely radial component of the metric is much more involved, and we will only analyze the asymptotic case $R \rightarrow \infty$. According to our discussion at the end of Sec. 3, this asymptotic expectation value is equal to the limit of $\hat{h}_{RR}(\bar{R}, T|g, 1)$ when $\bar{R} \rightarrow \infty$. Employing Eq. (3.12) and the operator identities (A.5), one arrives at

$$\langle \hat{h}_{RR}(\bar{R}, T|g, 1) \rangle_C = \langle e^{\hat{\gamma}_\infty} \rangle_C e^{-\langle \hat{\psi}(\bar{R}, T|g) \rangle_C}. \quad (4.4)$$

Here, $\langle e^{\hat{\gamma}_\infty} \rangle_C$ is precisely the coherent expectation value obtained in three dimensions for the diagonal R component of the asymptotic metric [23]:

$$\langle e^{\hat{\gamma}_\infty} \rangle_C = e^{\int_0^\infty dk (e^k - 1) |C(k)|^2}. \quad (4.5)$$

Notice that Eq. (4.4) can be understood as the quantum counterpart of the classical relation $h_{RR} = e^{\gamma - \psi}$ when γ is set equal to its asymptotic value. In fact, this non-trivial result is due to the factor ordering adopted in Eq. (3.9). Furthermore, assuming that there exist strictly positive constants k_1 and α such that the function $g(k)k^{1/2-\alpha}$ is bounded in the interval $[0, k_1]$, we prove in Appendix B that the limit $R \rightarrow \infty$ of the right-hand side of Eq. (4.2) vanishes. Therefore, the asymptotic expectation value of \hat{h}_{RR} in a coherent state turns out to coincide then with $\langle e^{\hat{\gamma}_\infty} \rangle_C$. Once again, this coincidence can be interpreted as the analog of the classical identity $h_{RR}(R = \infty) = e^{\gamma_\infty}$, which incorporates the boundary condition that ψ vanish at infinity. Taking into account that the expectation value $\langle e^{\hat{\gamma}_\infty} \rangle_C$ equals the classical value of e^{γ_∞} (at least) if the wave profile $C(k)$ has negligible high-energy contributions [4], we conclude that all coherent states in the low-energy sector would admit an approximate classical description of the four-dimensional geometry in the asymptotic region provided that they have small relative fluctuations in the metric when $R \rightarrow \infty$.

Before continuing our analysis, let us briefly comment on the assumption introduced above about the real function $g \in L^2(\mathbb{R}^+, dk)$ employed in the regularization. The existence of a bound in an interval starting at the origin is clearly satisfied for the function g itself (i.e., with $\alpha = 1/2$) if $g(k)$ is a cut-off in momentum space; in that case, $g(k) \leq 1$ on the positive axis. In addition, if the adopted regularization can be interpreted as a smooth spatial smearing, the function $g(k)$ is bounded again on the whole semiaxis $k \geq 0$, because $g(\vec{k}) \equiv g(k = |\vec{k}|)$ given by Eq. (3.4) is a Schwartz test function in \mathbb{R}^2 . These facts strongly support our hypothesis and show its compatibility with a wide class of feasible regularizations.

As a first step in the calculation of the metric fluctuations in the asymptotic region, one can check that

$$\Xi_C \hat{h}_{\theta\theta}(\bar{R}, T|g) = \Xi_C \hat{h}_{ZZ}(\bar{R}, T|g) = e^{\|g_{\bar{R}}\|^2} - 1, \quad (4.6)$$

$$\Xi_C \hat{h}_{RR}(\bar{R}, T|g, 1) = e^{\|\check{C}\|^2} e^{-\langle \hat{\psi}(\bar{R}, T|g) \rangle_{\check{C}}} e^{\|g_{\bar{R}}\|^2} - 1, \quad (4.7)$$

where $\check{C}(k) = C(k)(e^k - 1)$ [24] and $\Xi_C \hat{a} = (\langle \hat{a}^2 \rangle_C / \langle \hat{a} \rangle_C^2) - 1$ is the square of the relative uncertainty in the operator \hat{a} for the coherent state $|C\rangle$. It is worth noticing that, when $g = 0$, Eq. (4.7) reproduces the asymptotic fluctuations in the radial component of the three-metric studied by Ashtekar [4, 23]. In order to deduce the value of the asymptotic fluctuations, one only needs to take the limit $\bar{R} \rightarrow \infty$ in the above expressions. Actually, with our assumption about the existence of a segment to the right of the origin where the function $g(k)k^{1/2-\alpha}$ is bounded for some choice of $\alpha > 0$, it is shown in Appendix B that the asymptotic limits of $\|g_{\bar{R}}\|$ and $\langle \hat{\psi}(\bar{R}, T|g) \rangle_{\check{C}}$ vanish. Hence, for any of the coherent states, all metric operators display a classical behavior in the asymptotic region, except the operator that describes the purely radial component. Moreover, the square of the relative uncertainty in this last operator is given by the quantity $e^{\|\check{C}\|^2} - 1$, which is precisely the value of the corresponding uncertainty in the three-dimensional Einstein-Maxwell analog of our cylindrical system [4].

As a straightforward consequence, it turns out that all the results reached by Ashtekar in three dimensions about the appearance of large quantum gravity effects apply as well to the four-dimensional model constructed here for the Einstein-Rosen waves. Indeed, the conclusions reached by Ashtekar are not only qualitatively valid from a four-dimensional point of view, but also quantitatively accurate. The only existing difference is that, as far as the four-metric is concerned, one does not need to demand that the relative fluctuations in the basic field ψ (and in the physical quantities associated with it, like, e.g., the Hamiltonian) be negligible. Then, it is not necessary that the coherent states contain a large number of elementary excitations, a condition that is imposed in the three-dimensional system [4]. From this perspective, there exist more coherent states that admit a classical description of the four-metric in the asymptotic region than those that provide a meaningful semiclassical solution to the Einstein-Maxwell model obtained by dimensional reduction.

Summarizing, in order for the classical approximation to be acceptable in

the asymptotic region only two conditions must be verified [24]:

$$\int_0^\infty dk |C(k)|^2 (e^k - 1 - k) \ll 1, \quad \int_0^\infty dk |C(k)|^2 (e^k - 1)^2 \ll 1. \quad (4.8)$$

The first condition ensures that the coherent expectation value of $e^{\hat{\gamma}_\infty}$ coincides with the classical value of e^{γ_∞} . The second condition guarantees that the asymptotic fluctuations in the radial component of the metric are sufficiently small. In fact, the latter of these inequalities turns out to imply the former. In particular, for a wave profile $C(k)$ peaked around a certain wave number k_0 and with expected number of “particles” equal to $N = \int dk |C(k)|^2$ [4], the above conditions reduce to $N(e^{k_0} - 1)^2 \ll 1$.

Finally, let us notice that Eq. (4.6) determines the metric fluctuations in the θ and Z components at all points of the spacetime, and not just in the asymptotic region. It is then possible to obtain a useful estimate of those fluctuations also on the symmetry axis $R = 0$, at least for a physically reasonable class of regularization functions g . Taking into account the definition of g_R given in Eq. (3.6) and that the Bessel function J_0 equals the unity at the origin, one can check that the square norm $\|g_R\|^2$ becomes equal to $\|g\|^2/2$ when one approaches the axis. Suppose then that we further demand that the real regularization function $g \in L^2(\mathbb{R}^+, dk)$ take on the constant unit value in an interval starting at $k = 0$. This interval will have the generic form $[0, k_c]$, where k_c is a positive but otherwise arbitrary parameter. Notice that, in this case, one can make $k_1 \geq k_c$ and $\alpha = 1/2$, because the function g is bounded in an interval containing $[0, k_c]$. More importantly, according to our discussion in Sec. 3, all cut-off functions satisfy our new condition, with the parameter k_c being the cut-off introduced in momentum space. One can then interpret every function g in the considered family as a kind of generalized cut-off. Besides, it is clear that the regularization can still be viewed as a smooth spatial smearing if, in addition, $g(\vec{k}) \equiv g(k = |\vec{k}|)$ belongs to $\mathcal{S}(\mathbb{R}^2)$. For this class of regularization functions, one readily obtains that $\|g\|^2 \geq k_c$, so that, on the axis,

$$\Xi_C \hat{h}_{\theta\theta} = \Xi_C \hat{h}_{ZZ} \geq e^{k_c/2} - 1. \quad (4.9)$$

The relative uncertainties in the diagonal θ and Z components of the metric will thus become relevant on the symmetry axis unless $k_c \ll 1$. However, one would expect that, in our model, a physically reasonable (generalized) cut-off parameter k_c should be at least of the order of the inverse of the natural length scale provided by the system, i.e., $c^3/(\hbar G_3) \equiv k_P$ (in a general

system of units). With our conventions, $c = \hbar = 8G_3 = 1$, and thus $k_P = 8$. But, for $k_c \geq k_P = 8$, we get from Eq. (4.9) that $\Xi_C \hat{h}_{\theta\theta} = \Xi_C \hat{h}_{ZZ} > 50$. So, quantum gravity effects are huge on the symmetry axis for all of the considered regularizations and, therefore, also in the limit in which the cut-off is removed. In particular, this fact seems to indicate that the requirement of regularity on the axis of rotational symmetry is meaningless from a quantum mechanical point of view.

5 Conclusions

We have constructed a complete quantum theory that describes the metric of the family of Einstein-Rosen waves. This theory is based on the quantization carried out in Ref. [1] for the Einstein-Maxwell model obtained by the dimensional reduction of linearly polarized cylindrical gravity.

We have started with the Hamiltonian formulation of general relativity for spacetimes that admit two commuting spacelike Killing vectors. Introducing suitable gauge-fixing conditions adapted to cylindrical symmetry, we have been able to remove all the gravitational constraints. In this way, we have arrived at a reduced model for the most general family of cylindrical waves in vacuum gravity. We have also calculated the symplectic structure induced from general relativity and the Hamiltonian that generates the time evolution. This Hamiltonian has been computed by reducing the gravitational Einstein-Hilbert action supplemented with appropriate surface terms. Such terms include the contribution of the timelike boundary located at $R \rightarrow \infty$ (where R is the radial coordinate), and have been normalized to vanish in Minkowski spacetime.

We have then imposed the requirement of linear polarization as a symmetry condition. This has led to a reduced midisuperspace model whose classical solutions are precisely the Einstein-Rosen waves. The model has only one degree of freedom in configuration space, given by a cylindrically symmetric field ψ , and is indeed classically equivalent to a rotationally symmetric, massless scalar field (dual to a Maxwell field) coupled to three-dimensional gravity. The non-zero components of the four-metric in our reduced model are exponentials of the basic field ψ multiplied either by trivial functions or by the purely radial component of the three-metric in the Einstein-Maxwell system. Employing the quantum theory proposed in Ref. [1] for this three-dimensional model, we have then achieved a full quantization of the metric

for Einstein-Rosen waves. Since, using a Fock space representation in which the field ψ is represented as an operator-valued distribution, Ashtekar and Pierri had already succeeded in constructing a (presumably [3]) positive operator for the diagonal radial component of the metric in three dimensions, our quantization process has been reduced, basically, to the following two steps. Firstly, we have regularized the field ψ to reach a well-defined operator and avoid ultraviolet divergences. Secondly, owing to the non-linearity of the metric in ψ , we have introduced a reasonable choice of factor ordering for the metric operators.

We have also analyzed whether there exist large quantum gravity effects in the system, as happens to be the case in the Einstein-Maxwell counterpart of the model. We have shown that, with the chosen factor ordering, the expectation values of the diagonal θ and Z components of the four-metric correspond in fact to classical trajectories in all of the coherent states of the field ψ . In addition, we have seen that, like in the three-dimensional model, the asymptotic expectation value of the radial component is that predicted by the classical theory (semiclassical theory in three dimensions), provided that the coherent state has negligible contributions from the high-energy sector. In the derivation of this result, we have introduced the very weak assumption that the function $g \in L^2(\mathbb{R}^+, dk)$, employed in the regularization of the field ψ , is bounded in a certain interval around $k = 0$ when multiplied by a factor of the form $k^{1/2-\alpha}$, with α being a positive constant. Such an assumption is satisfied by all cut-off regularizations, as well as by those regularizations that can be interpreted as a smooth spatial smearing of the field, and therefore implies no restriction in physically relevant situations.

For such regularizations we have also computed the value of the quantum uncertainties in the metric when one approaches the asymptotic region. The fluctuations in the θ and Z components turn out to vanish when $R \rightarrow \infty$. Therefore, these metric operators display a classical asymptotic behavior. As far as they are concerned, the boundary condition that the basic field vanish asymptotically is respected quantum mechanically in all coherent states. The asymptotic fluctuations in the diagonal radial component, on the other hand, are exactly the same as in the three-dimensional model [4]. These results prove the validity of the analysis carried out by Ashtekar in three dimensions, not only qualitatively, but also quantitatively. There is only one caveat: in order for the four-dimensional metric to admit a classical description, it is not needed that the physical quantities associated with the field ψ possess small relative uncertainties. The requirement that the number of

fundamental excitations contained in the coherent state be large, necessary for a meaningful semiclassical approximation in the three-dimensional model [4], is no longer present. In this sense, the set of coherent states that are peaked around a classical four-metric in the asymptotic region is bigger than that corresponding to acceptable semiclassical solutions in three dimensions. In particular, the vacuum is contained only in the former of these sets. Of course, apart from the quantum metric, one could be interested in considering other operators related with the four-dimensional geometry (like those which describe the spacetime curvature). Demanding that such operators have negligible asymptotic fluctuations might well impose further conditions on the family of coherent states that display a classical behavior and, perhaps, restrict again their number of “particles”.

We have also analyzed the quantum fluctuations in the diagonal θ and Z components of the metric when one approaches the symmetry axis. We have shown that, for all regularizations that do not modify the mode decomposition of the field ψ up to wave numbers of the order of the natural scale $k_P = c^3/(\hbar G_3)$, the relative uncertainties in the metric at $R = 0$ are large. In particular, they explode in the limit in which the regularization is removed. One should then expect significant quantum effects on the axis. It is thus unclear up to what extent the condition of regularity of the four-geometry on the symmetry axis is sensible from a quantum mechanical perspective.

Although there exist other possible quantizations of our model, the quantum theory constructed presents clear advantages. In fact, it has been constructed in such a way that the relation between the metric operators in three and four dimensions are as simple and natural as possible. This fact has allowed us to compare the physical results for the Einstein-Rosen waves with those obtained by Ashtekar in the three-dimensional Einstein-Maxwell model, and prove that the latter are indeed relevant in four dimensions. Regarding the factor ordering, we have checked that the existence of large quantum fluctuations in the metric is rather insensitive to the operator ordering. However, it generally affects the expectation value of the metric in coherent states, so that, for factor orderings other than the one selected, such value would only reproduce a classical solution in the limit $\hbar \rightarrow 0$. Obviously, this is one of the reasons that motivated our choice of ordering.

On the other hand, it is worth noticing that our discussion about the expectation value of the metric and its uncertainty in the asymptotic region is in fact regularization independent, apart from the more than reasonable hypothesis that the regularization function (possibly multiplied by a factor

$k^{1/2-\alpha}$, with $\alpha > 0$) be bounded in a neighborhood of the origin of wave numbers, an assumption that, as we have commented, involves no physical limitation in practice. Our analysis of the metric uncertainties on the symmetry axis, nevertheless, has been restricted to a particular (though quite general) family of regularizations, which can be interpreted as a generalized cut-off. In this sense, our results about the fluctuations on the axis depend on the regularization adopted. However, since those fluctuations are always significant when the regularization is removed at scales below the inverse-length parameter k_P , naturally provided by the system, one would expect the existence of important quantum gravity effects on the axis of cylindrical symmetry in all physically plausible situations. Finally, since coherence in the basic field ψ is not a requisite for the validity of the classical approximation from a purely four-dimensional viewpoint, it would be interesting to investigate whether the quantum fluctuations in the four-metric can be diminished by considering other families of quantum states, like, e.g., those analyzed by Gambini and Pullin [5].

Acknowledgments

M. E. A. acknowledges the financial support provided by C.S.I.C. during the completion of this work. G. A. M. M. was supported by DGEIC under the Research Projects No. PB97-1218 and HP1988-0040.

Appendix A

In this appendix, we will prove relation (3.12). We will make use of the operator expansion theorem [21]

$$e^{x\hat{a}} \hat{b} e^{-x\hat{a}} = \sum_{n=0}^{\infty} \frac{x^n}{n!} [\hat{a}, \hat{b}]_{(n)} \quad (\text{A.1})$$

and the identity [21]

$$e^{\hat{a}} e^{\hat{b}} e^{-\hat{a}} = \exp \left(e^{\hat{a}} \hat{b} e^{-\hat{a}} \right). \quad (\text{A.2})$$

In these expressions, \hat{a} and \hat{b} denote two generic operators and $[\hat{a}, \cdot]_{(n)}$ is the n -th application of the commutator with \hat{a} . Particularizing these equations to the case in which $\hat{b} =: \hat{\gamma}_{\infty} :$, $x = 1$, and

$$\hat{a} = \frac{1}{\sqrt{2}} \int_0^{\infty} \frac{dk}{k} J_0(k\bar{R}) g_1(k) \left[\hat{A}^{\dagger}(k) e^{ikT} - \hat{A}(k) e^{-ikT} \right] \equiv \hat{D}(\bar{R}, T|g), \quad (\text{A.3})$$

we obtain

$$\hat{h}_{RR}(\bar{R}, T|g, 1) = e^{-\|\check{g}_{\bar{R}}\|^2} e^{\hat{D}(\bar{R}, T|g)} e^{:\hat{\gamma}_{\infty}:} e^{-\hat{D}(\bar{R}, T|g)}. \quad (\text{A.4})$$

Here, $\check{g}_R(k) = g_R(k)/\sqrt{e^k - 1}$ and we have employed definitions (3.6) and (3.10). On the other hand, a repeated application of the operator expansion theorem to calculate the commutator of $e^{:\hat{\gamma}_{\infty}:}$, firstly with the smeared version of the creation and annihilation operators, and then with their exponentials, leads to

$$\begin{aligned} e^{:\hat{\gamma}_{\infty}:} e^{\int_0^\infty dk f(k) \hat{A}^\dagger(k)} &= e^{\int_0^\infty dk f(k) e^k \hat{A}^\dagger(k)} e^{:\hat{\gamma}_{\infty}:}, \\ e^{\int_0^\infty dk f(k) \hat{A}(k)} e^{:\hat{\gamma}_{\infty}:} &= e^{:\hat{\gamma}_{\infty}:} e^{\int_0^\infty dk f(k) e^k \hat{A}(k)}. \end{aligned} \quad (\text{A.5})$$

Using these relations, together with the CBH formula, one can readily check that the right-hand sides of Eqs. (3.12) and (A.4) coincide.

Appendix B

We want to prove that the expectation value $\langle \hat{\psi}(R, T|g) \rangle_C$ and the norm $\|g_R\|$ vanish in the asymptotic limit $R \rightarrow \infty$ if the functions C and g belong to the Hilbert space $L^2(\mathbb{R}^+, dk)$ and, for some choice of positive constant α , the function $g(k)k^{1/2-\alpha}$ is bounded in an interval of the form $[0, k_1]$. Here, k_1 is a strictly positive number and $g_R(k) = J_0(kR)g(k)/\sqrt{2}$. In fact, we only need to show that $\|g_R\|$ vanishes in the asymptotic region, because, using Eq. (4.2), the triangle inequality for complex numbers, and the Schwarz inequality on $L^2(\mathbb{R}^+, dk)$, one gets

$$|\langle \hat{\psi}(R, T|g) \rangle_C| \leq 2\|C\| \|g_R\|. \quad (\text{B.1})$$

Obviously, the same arguments apply to the value of $\langle \hat{\psi}(R, T|g) \rangle_{\check{C}}$ appearing in Eq. (4.7). Let us then write

$$\|g_R\|^2 = \frac{1}{2} \int_0^{k_1} dk J_0^2(kR) |g(k)|^2 + \frac{1}{2} \int_{k_1}^\infty dk J_0^2(kR) |g(k)|^2. \quad (\text{B.2})$$

The second term on the right-hand side vanishes when $R \rightarrow \infty$ because, in that limit, $J_0^2(kR)$ tends to zero uniformly in $k \in [k_1, \infty)$, with $k_1 > 0$. As for the first term, let G be the upper bound of $|g(k)k^{1/2-\alpha}|$ in $[0, k_1]$. Then

$$\frac{1}{2} \int_0^{k_1} dk J_0^2(kR) |g(k)|^2 \leq \frac{G^2}{2R^{2\alpha}} \int_0^{k_1 R} \frac{dk}{k^{1-2\alpha}} J_0^2(k). \quad (\text{B.3})$$

Recalling that $\alpha > 0$ and $J_0(k) \approx \cos(k - \pi/4)\sqrt{2/(\pi k)}$ (up to subdominant terms) for $k \gg 1$, one can finally show that the limit of the above expression when $R \rightarrow \infty$ is zero.

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